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ON COMPACT COHOMOLOGY THEORIES AND PONTRJAGIN DUALITY

BY

KEITH JOHNSON

ABSTRACT. Cohomology theories taking values in the category of topological groups are examined and a representation theorem is established for those whose coefficient groups are compact and locally euclidean. A method for constructing unstable homology operations is developed using this theorem, and application is made to the case of complex K-theory.

Introduction. The idea of a cohomology theory whose values are compact topological groups is not a new one in algebraic topology. Eilenberg and Steenrod considered this possibility for ordinary cohomology in [**6**], while the generalized case has been considered in [**5**, **14** and **17**]. The present paper is devoted to the study of theories of this sort with the restriction that their coefficient groups are locally euclidean.

One reason for considering these theories is their relation to homology, arising from Pontrjagin duality. If c is the Pontrjagin character functor, $\mathrm{Hom}_c(\ ,\mathbf{R}/\mathbf{Z})$, and $E_*(\ )$ is a generalized homology theory, then $c(E_*(\ ))$ is a compact cohomology theory. Its coefficients are locally euclidean iff the coefficients of $E_*(\ )$ are finitely generated.

According to E. H. Brown's representation theorem [**4**], a generalized cohomology theory, $h^*(\ )$, is represented by a spectrum, E, in the sense that there is a natural equivalence $h^*(\ ) \simeq [\ , E]_*$. Thus the problem in representing a compact cohomology theory is in representing the topology on the groups involved.

Denote by $\mathbb{S}$ the category of spectra, as constructed in [**15**], and by $\mathbb{S}_{fg}$ the full subcategory of spectra whose homotopy groups are finitely generated. For $E \in \mathbb{S}$ we will denote by $E\mathbf{R}/\mathbf{Z}$ the spectrum representing E-cohomology with $\mathbf{R}/\mathbf{Z}$ coefficients as constructed in [**8**]. Our results are:

THEOREM 1. *If $E \in \mathbb{S}_{fg}$, then there is a canonical way of making $E\mathbf{R}/\mathbf{Z}^*(\ )$ into a compact cohomology theory.*

We will denote $E\mathbf{R}/\mathbf{Z}^*(\ )$ equipped with this extra structure by $E\mathbf{R}/\mathbf{Z}^*_T(\ )$.

THEOREM 2. *If $h^*(\ )$ is a compact cohomology theory with locally euclidean coefficient groups, then there exists a spectrum $E \in \mathbb{S}_{fg}$, unique up to homotopy equivalence, such that $h^*(\ ) = E\mathbf{R}/\mathbf{Z}^*_T(\ )$.*

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To establish these theorems we must first examine the category of compact locally euclidean groups. In the first part of §1, we establish the necessary result, expressed as a natural equivalence of the category of locally euclidean compact groups with a new category which we construct. The second part of §1 deals with locally rational groups and is not necessary for the proofs of Theorems 1 and 2. It is used in §5 for some applications.

§2 recalls from [**5**] the basic results about Pontrjagin duality of spectra while §3 recalls from [**2**] results about another duality functor on the category of spectra which was defined by D. W. Anderson. The idea behind the proof of our theorems is that Pontrjagin and Anderson duality of spectra are equivalent. We make this precise in §4.

In §5 we present two applications of our representation theorem. The first is concerned with unstable homology operations. These are natural transformations $E_n(\ ) \to E_m(\ )$ where $E_*(\ )$ is a generalized homology theory. We produce a method of constructing such operations from unstable cohomology operations and apply it to the case of complex K-theory.

The results are:

THEOREM 3. (a) *If K denotes the spectrum representing complex K-theory then the Pontrjagin dual of $K_*(\ )$ is $K\mathbf{R}/\mathbf{Z}^*_T(\ )$.*

(b) *There exist homology operations $K_0(\ ) \to K_0(\ )$ dual to the Adams operations.*

The second application is the determination of the role of the topology on the values of a compact cohomology theory. More precisely, if F is the forgetful functor which ignores the topology on a topological group and if $h^*(\ )$ is a compact cohomology theory, is $h^*(\ )$ determined by $F \circ h^*(\ )$? In [**5**] much use was made of the fact that the answer is yes if the coefficient groups of $h^*(\ )$ are all finite. We will show

THEOREM 4. *If h, h' are compact cohomology theories with locally euclidean coefficients, and E, E' are the spectra associated to them as in Theorem 2, then the following are equivalent:*

(a) $F \circ h = F \circ h'$,

(b) $E\mathbf{R}/\mathbf{Z} \simeq E'\mathbf{R}/\mathbf{Z}$,

(c) $EQ/\mathbf{Z} \simeq E'Q/\mathbf{Z}$,

(d) $\hat{E} \simeq \hat{E}'$.

Here $\hat{E}$ denotes the profinite completion of E in the sense of Sullivan [**14**].

COROLLARY 5. *There exist two compact cohomology theories with locally euclidean coefficient groups h, h' such that $F \circ h = F \circ h'$ but $h \neq h'$.*

1. Locally euclidean groups.

This section is devoted to studying the category AB_{LE} of locally euclidean compact abelian groups. We will construct three other categories, each equivalent to AB_{LE}, and will produce functors effecting these equivalences. These new categories represent the topology on objects of AB_{LE} in a form that is particularly convenient for the study of compact cohomology theories.

DEFINITION 6. AB_{LE} is the full subcategory of the category of topological groups consisting of those groups which are compact, hausdorff and abelian, and which have a neighbourhood of 0 homeomorphic to $\mathbf{R}^n$ for some integer n.

DEFINITION 7. C_{LE} is the category whose objects are triples (V, G, p), where V is a finite-dimensional $\mathbf{R}$-vector space, G is an abelian group, and $p\colon V \to G$ is a group homomorphism with the properties:

(1) $\mathrm{Ker}(p)$ is a lattice subgroup of V,

(2) $\mathrm{Coker}(p)$ is a finite quotient of G.

The morphisms in C_{LE} are pairs (f_1, f_2) where f_2 is a homomorphism, f_1 is $\mathbf{R}$-linear, and

$$\begin{array}{ccc} V & \xrightarrow{f_1} & V' \\ p\downarrow & & p'\downarrow \\ G & \xrightarrow{f_2} & G' \end{array}$$

commutes.

We define functors $F\colon AB_{\mathrm{LE}} \to C_{\mathrm{LE}}$, $G\colon C_{\mathrm{LE}} \to AB_{\mathrm{LE}}$ giving an equivalence between these categories as follows:

DEFINITION 8. For $G \in AB_{\mathrm{LE}}$, let $L(G)$ denote the set of 1-parameter subgroups of G, i.e. continuous homomorphisms from $\mathbf{R}$ to G. $L(G)$ has a natural $\mathbf{R}$ vector space structure given by

$$(\alpha f)(t) = f(\alpha t), \qquad (f+g)(t) = f(t) + g(t).$$

Also, let $\bar{p}\colon L(G) \to G$ via $\bar{p}(f) = f(1)$. We define F by $F(G) = (L(G), G, \bar{p})$.

REMARK. Those familiar with the solution of Hilbert's 5th problem will recognize $L(G)$ as the Lie algebra constructed to show that G is a Lie group. (In the abelian case the Lie product is trivial.)

DEFINITION 9. A finite-dimensional $\mathbf{R}$-vector space has a unique topology making it a topological vector space [**11**, p. 21]. For $(V, G, p) \in C_{\mathrm{LE}}$, let $G(V, G, p)$ be the abelian group G equipped with the strongest topology making p continuous.

LEMMA 10. *$F(G) = (L(G), G, \bar{p})$ is an object in C_{LE}.*

PROOF. The structure theorem for compact locally euclidean abelian groups [**10**, p. 104] implies that G is isomorphic to $T^n \times F$, where T^n is the n-torus and F is a finite abelian group. It is straightforward to verify that $L(T^n \times F) = \mathbf{R}^n$ and $\bar{p}$ has the required properties.

LEMMA 11. *$G(V, G, p)$ is an object in AB_{LE}.*

PROOF. Since $\ker p$ is a lattice, $V/\ker(p)$ is compact and so $G(V, G, p)$ is an extension of a compact group by a finite one, hence compact. Also because $\ker(p)$ is a lattice, p gives a local homeomorphism between γ and $G(V, G, p)$. Thus $G(V, G, p)$ is locally euclidean and hausdorff.

THEOREM 12. *F and G give an equivalence of categories.*

PROOF. The composite $G \circ F$ is the identity functor on AB_{LE}.

A natural transformation from the identity functor of C_{LE} to $F \circ G$ is given as follows: for $(V, G, p) \in C_{\mathrm{LE}}$ define a map (f_1, f_2): $(V, G, p) \to (L(G), G, \bar{p})$ by $f_1(v)(t) = p(t \cdot v), f_2 = \mathrm{id}$.

The category C_{LE} and Theorem 12 provide the information about locally euclidean groups needed to establish our representation theorem for compact cohomology theories. For calculations, locally euclidean groups are unnecessarily large, however. A more tractable substitute is given by the category AB_{LR} of "locally rational" groups which we now consider.

DEFINITION 13. AB_{LR} is the full subcategory of the category of topological groups consisting of those groups which are the torsion subgroups of objects of AB_{LE} equipped with the subgroup topology.

DEFINITION 14. C_{LR} is the category whose objects are triples (V, G, p), where V is a finite-dimensional Q-vector space, G is an abelian group, and p: $V \to G$ is a group homomorphism with the properties:

(1) $\mathrm{Ker}(p)$ is a lattice subgroup of V;

(2) $\mathrm{Coker}(p)$ is a finite quotient of G.

The morphisms in C_{LR} are pairs (f_1, f_2), where f_2 is a group homomorphism, f_1 is Q linear, and

$$\begin{tikzcd} V \arrow[r, "f_1"] \arrow[d] & V' \arrow[d] \\ G \arrow[r, "f_2"] & G \end{tikzcd}$$

commutes.

As in the euclidean case we will define functors F', G' between AB_{LR} and C_{LR} giving an equivalence of categories. To do this it is convenient to first establish the relation between the locally euclidean and locally rational cases.

DEFINITION 15. For $G \in AB_{\mathrm{LE}}$, let $T(G)$ denote the torsion subgroup of G with the subspace topology. For $G \in AB_{\mathrm{LR}}$ let $S(G)$ denote the completion of G [**7**, p. 69].

PROPOSITION 16. *S and T are functors between AB_{LE} and AB_{LR} and give an equivalence of categories.*

PROOF. It follows from the structure theorem for compact locally euclidean groups [**10**, p. 104] that the torsion subgroup of such a group is dense. Thus S is well defined.

Suppose $G \in AB_{\mathrm{LE}}$ and let i: $T(G) \to G$ be the inclusion. Since G is complete, i extends uniquely to a map $\bar{i}$: $ST(G) \to G$ which is an isomorphism, since $T(G)$ is a dense subgroup.

Conversely, suppose $G \in AB_{\mathrm{LR}}$. Since G is a torsion group the inclusion $G \to S(G)$ has image in the torsion subgroup and so gives a map $G \to TS(G)$. The structure theorem implies that this is an isomorphism.

DEFINITION 17. If $(V, G, p) \in C_{\mathrm{LE}}$, let G_T be the torsion subgroup of G and define (V_T, G_T, p_T) by the pull-back square

$$\begin{tikzcd} V_T \arrow[r] \arrow[d, "p_T"] & V \arrow[d, "p"] \\ G_T \arrow[r] & G \end{tikzcd}$$

This defines a functor $T'\colon C_{\mathrm{LE}} \to C_{\mathrm{LR}}$.

DEFINITION 18. If $(V, G, p) \in C_{\mathrm{LR}}$, let $\bar{V}$ be the completion of V (which is an $\mathbf{R}$-vector space) and define $(\bar{V}, \bar{G}, \bar{p})$ by the push-out square

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p"] & \bar{V} \arrow[d, "\bar{p}"] \\ G \arrow[r] & \bar{G} \end{tikzcd}$$

This defines a functor $S'\colon C_{\mathrm{LR}} \to C_{\mathrm{LE}}$.

PROPOSITION 19. *The functors S' and T' give an equivalence of categories.*

PROOF. To define a natural transformation from $S' \circ T'$ to the identity functor of C_{LE}, note that $S' \circ T'(V, G, p)$ is contained in the diagram

$$\begin{tikzcd} & & V \arrow[ddd] \\ V_T \arrow[r] \arrow[urr] \arrow[d] & \bar{V} \arrow[d] & \\ G_T \arrow[r] \arrow[drr] & \bar{G} & \\ & & G \end{tikzcd}$$

Since V is complete there is a unique map $\bar{V} \to V$ fitting into this diagram. By the universal property of the push-out, there is a unique homomorphism $\bar{G} \to G$ fitting into this diagram. Together these two maps give the required morphism $(\bar{V}, \bar{G}, \bar{p}) \to (V, G, p)$. The verification that this is an isomorphism is straightforward.

In a similar way, using the universal property of the pull-back we may define a natural equivalence from $T' \circ S'$ to the identity functor of C_{LR}.

If we let $F' = T' \circ F \circ S$ and $G' = T \circ G \circ S'$ then we have

THEOREM 20. *The functors F' and G' give an equivalence between the categories AB_{LR} and C_{LR}.*

A direct description of these two functors is given as follows.

DEFINITION 21. For $G \in AB_{\mathrm{LR}}$, let $L'(G)$ denote the Q vector space of homomorphisms from Q to G. Let $p'\colon L'(G) \to G$ be the homomorphism $p'(f) = f(1)$. Define $F''(G) = (L'(G), G, p')$.

DEFINITION 22. For $(V, G, p) \in AB_{\mathrm{LR}}$, V is a finite-dimensional Q vector space and has a unique topology making it a topological vector space. Let $G'((V, G, p)) = g$ equipped with the strongest topology which makes p continuous.

PROPOSITION 23. *The functors F' and F'' are naturally equivalent as are the functors G' and G''.*

2. Pontrjagin duality. In this section we recall the definition and basic properties of the Pontrjagin dual of a spectrum. $c(G) = \mathrm{Hom}_c(G, \mathbf{R}/\mathbf{Z})$ denotes the Pontrjagin character functor which gives a duality between discrete and compact abelian groups. It takes short exact sequences to short exact sequences and direct sums to direct products. Thus, if $E_*(\ )$ is a generalized homology theory, $c(E_*(\ ))$ will be a cohomology theory taking values in the category of compact abelian groups. Ignoring the topology on the groups, a spectrum E' may be found representing this cohomology theory. The correspondence $E \to E'$ is made functorial as follows:

Choose a spectrum $c(S)$ and a natural equivalence t_s: $(c(S))^*(\ ) \simeq c(S_*(\ ))$ where S is the sphere spectrum.

DEFINITION 24. A Pontrjagin duality between spectra E and E' is a pairing μ: $E \wedge E' \to c(S)$ such that the Kronecker index $E_q(X) \otimes (E')^q(x) \to c(S)^0(S) = \mathbf{R}/\mathbf{Z}$ induces an isomorphism $T_\mu(E')^q(x) \simeq c(E_q(x))$ for all q and all CW complexes X.

THEOREM 25 [5]. *E' is a Pontrjagin dual of E iff E' represents the function spectrum $F(E, c(S))$.*

Thus for each spectrum E we may choose a Pontrjagin dual $c(E)$ and a duality pairing μ_E: $E \wedge c(E) \to c(S)$. If f: $E \to F$ is a morphism of spectra, then we may take $c(f)$ to be the dual of f with respect to μ_E and μ_F. We thus have

THEOREM 26 [5]. *c: $\mathcal{S} \to \mathcal{S}$ is a contravariant functor.*

This definition has ignored the topology on the groups involved, so it is unreasonable to expect c to be a duality on the category of spectra.

DEFINITION 27. Let $\mathcal{S}_f$ denote the category of spectra all of whose homotopy groups are finite.

Since a finite abelian group has a unique topology making it a compact group, namely the discrete topology, Brown and Comenetz were able to show

THEOREM 28 [5]. *$c \circ c$: $\mathcal{S}_f \to \mathcal{S}_f$ is naturally equivalent to the identity functor.*

3. Anderson duality. In [2] Anderson defined a functor D on the category $\mathcal{S}_{fq}$ of spectra with finitely generated homotopy groups. The definition was

DEFINITION 29. For $E \in \mathcal{S}_{fq}$ let E_Q, $E_{Q/\mathbf{Z}}$ be spectra representing the cohomology theories $\mathrm{Hom}(E_*(\ ), Q)$ and $\mathrm{Hom}(E_*(\ ), Q/\mathbf{Z})$, respectively, and let q_E: $E_Q \to E_{Q/\mathbf{Z}}$ be a map of spectra representing the natural transformation induced by the projection $Q \to Q/\mathbf{Z}$. DE is the cofibre of q_E.

To make this definition functorial we follow the method of Brown and Comenetz. If G is any injective abelian group, $\mathrm{Hom}(E_*(\ ), G)$ is a generalized cohomology theory and so can be represented as the case $G = \mathbf{R}/\mathbf{Z}$ was in §2.

Choose spectra C_QS, $C_{Q/\mathbf{Z}}S$, $C_{\mathbf{R}}S$, and maps of spectra

$$(*) \qquad \begin{array}{ccc} C_QS & \to & C_{\mathbf{R}}S \\ \downarrow q_s' & & \downarrow q_s \\ C_{Q/\mathbf{Z}}S & \to & C_{\mathbf{R}/\mathbf{Z}}S = c(S) \end{array}$$

representing the cohomology theories and natural transformation in the diagram

$$\begin{array}{ccc} \mathrm{Hom}(S_*(\ ), Q) & \to & \mathrm{Hom}(S_*(\ ), \mathbf{R}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(S_*(\ ), Q/\mathbf{Z}) & \to & \mathrm{Hom}(S_*(\ ), \mathbf{R}/\mathbf{Z}) \end{array}$$

Let $D'S$, DS be the cofibres of the vertical maps of $(*)$ and choose a map $D'S \to DS$ completing $(*)$ to a map of exact triangles [**15**, p. 170].

LEMMA 30. *This map is a homotopy equivalence.*

PROOF. This follows from the fact that it induces an isomorphism of homotopy groups.

DEFINITION 31. An Anderson duality between spectra E and E' in $\mathfrak{S}_{fg}$ is a pairing μ: $E \wedge E' \to DS$ such that the natural transformation t_μ: $[X, E'] \to [E \wedge X, DS]$ given by $t_\mu(f) = \mu \circ (1 \wedge f)$ is a natural equivalence.

From the definition of function spectra [**15**, p. 194] we now have

THEOREM 32. *E' is an Anderson dual of E iff E' represents the function spectrum $F(E, DS)$.*

D may now be made into a functor just as c was in §2. The essential property of D is

THEOREM 33. *$D \circ D$ is naturally equivalent to the identity functor of $\mathfrak{S}_{fg}$.*

Only an outline of the proof of this theorem was given in [**2**]. A complete version may be found in [**16**, p. 210].

4. Proof of the main theorems. Recall [**8**] the process of introducing coefficients into a cohomology theory:

DEFINITION 34. If E is a spectrum and G an abelian group, let $EG = E \wedge MG$, where MG is a Moore spectrum, i.e. a spectrum with the properties:

$$\Pi_i(MG) = 0, \quad i < 0, \qquad \Pi_0(MG) = H_0(MG) = 0, \qquad H_i(MG) = 0, \quad i > 0.$$

Choose spectra $M\mathbf{R}$, $M\mathbf{R}/\mathbf{Z}$ and a map p: $M\mathbf{R} \to M\mathbf{R}/\mathbf{Z}$ whose action in $H_0(\ )$ is the projection $\bar{p}$: $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$.

The values of the cohomology theory $EG^*(\ )$ are related to those of $E^*(\ )$ by the following universal coefficient theorem.

THEOREM 35 [**8**]. *For any spectrum E, abelian group G, and finite CW complex X there is an exact sequence*

$$0 \to E^n(X) \otimes G \to EG^n(X) \to \mathrm{Tor}(E^{n+1}(X), G) \to 0$$

natural with respect to E and X.

PROPOSITION 36. *For $E \in \mathfrak{S}_{fg}$, $(E\mathbf{R}^*(\ ), E\mathbf{R}/\mathbf{Z}^*(\ ), (1 \wedge p)^*)$ is a functor from the category of finite CW complexes to C_{LE}.*

PROOF. The universal coefficient theorem yields the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & E^*(X)\otimes \mathbf{R} & \to & E\mathbf{R}^*(X) & \to & 0 & & \\
 & & \downarrow 1\otimes\bar{p} & & \downarrow (1\wedge p)^* & & & & \\
0 & \to & E^*(X)\otimes \mathbf{R}/\mathbf{Z} & \to & E\mathbf{R}/\mathbf{Z}^*(X) & \to & \operatorname{Tor}\left(E^{*+1}(X),\mathbf{R}/\mathbf{Z}\right) & \to & 0
\end{array}$$

If X is a finite CW complex, $\operatorname{Tor}(E^{*+1}(X), \mathbf{R}/\mathbf{Z})$ is a finite group and $E^*(X)\otimes\mathbf{R} = E\mathbf{R}^*(X)$ is a finite-dimensional $\mathbf{R}$-vector space. From the commutativity of the diagram, $\ker((1\wedge p)^*) \simeq \ker(1\otimes\bar{p})$, which is a lattice subgroup. Thus $(E\mathbf{R}^*(X), E\mathbf{R}/\mathbf{Z}^*(X), (1\wedge p)^*)$ is an object in C_{LE}. That a continuous map of finite CW complexes induces a morphism in C_{LE} is straightforward.

PROOF OF THEOREM 1. If we apply the functor G of Definition 9 to $(E\mathbf{R}^*(\ ), E\mathbf{R}/\mathbf{Z}^*(\ ), (1\wedge p)^*)$ we obtain a topology on $E\mathbf{R}/\mathbf{Z}^*(\ )$, as required.

To establish Theorem 2 we must relate $DE\mathbf{R}/\mathbf{Z}$ and $c(E)$. Using the $\mathbf{R}$ vector space structure of $C_{\mathbf{R}}S^*(X)$ we may make $C_{\mathbf{R}}S$ into an $M\mathbf{R}$ module spectrum. Let $\mathcal{V}\colon C_{\mathbf{R}}S\wedge M\mathbf{R}\to C_{\mathbf{R}}S$ be the structure map for this module spectrum, and let $\alpha(S)$ be the composition $DS\wedge M\mathbf{R}\to C_{\mathbf{R}}S\wedge M\mathbf{R}\to C_{\mathbf{R}}S$. Also let $\beta(S)$ be the map completing the following diagram to a morphism of exact triangles [15, p. 170]:

$$\begin{array}{ccc}
DS & = & DS \\
\downarrow & & \downarrow \\
DS\wedge M\mathbf{R} & \xrightarrow{\alpha(S)} & C_{\mathbf{R}}S \\
\downarrow & & \downarrow \\
DS\wedge M\mathbf{R}/\mathbf{Z} & \xrightarrow{\beta(S)} & C_{\mathbf{R}/\mathbf{Z}}S
\end{array}$$

By calculating the action of $\alpha(S)$ on homotopy groups we see that $\alpha(S)$ is a homotopy equivalence. By the 5-lemma, $\beta(S)$ is also.

DEFINITION 37. For $E\in\mathbb{S}_{fg}$ define maps $\alpha(E)$, $\beta(E)$ in the diagram

$$\begin{array}{ccc}
DS & = & DE \\
\downarrow & & \downarrow \\
DE\wedge M\mathbf{R} & \xrightarrow{\alpha(E)} & C_{\mathbf{R}}E \\
\downarrow & & \downarrow \\
DE\wedge M\mathbf{R}/\mathbf{Z} & \xrightarrow{\beta(E)} & C_{\mathbf{R}/\mathbf{Z}}E
\end{array}$$

to be the adjoints of

$$E\wedge DE\wedge M\mathbf{R}\xrightarrow{\mu\wedge 1} DS\wedge M\mathbf{R}\xrightarrow{\alpha(S)} C_{\mathbf{R}}S$$

and

$$E\wedge DE\wedge M\mathbf{R}/\mathbf{Z}\xrightarrow{\mu\wedge 1} DS\wedge M\mathbf{R}/\mathbf{Z}\xrightarrow{\beta(S)} C_{\mathbf{R}/\mathbf{Z}}S.$$

PROPOSITION 38. $((E)^*, (E)^*)$ *gives an equivalence between the functors* $(E\mathbf{R}^*(\ ),$ $E\mathbf{R}/\mathbf{Z}^*(\ ), (1 \wedge p)^*)$ *and* $(C_{\mathbf{R}}E^*(\ ), C_{\mathbf{R}/\mathbf{Z}}C^*(\ ), q_E^*)$ *taking values in* C_{LE}.

PROOF. It suffices to show that $\alpha(E)$ is a homotopy equivalence since, by the 5-lemma, $\beta(E)$ will be also. This can be verified by computing the action on homotopy groups.

COROLLARY 39. $\mathrm{Hom}_c(E_*(\ ), \mathbf{R}/\mathbf{Z})$ *and* $DE\mathbf{R}/\mathbf{Z}_T^*(\ )$ *are equivalent as compact cohomology theories.*

PROOF OF THEOREM 2. Since $h^*(\ )$ is a compact cohomology theory, $\mathrm{Hom}_c(h^*(\ ), \mathbf{R}/\mathbf{Z})$ is discrete. Let F be the spectrum representing $\mathrm{Hom}_c(h^*(\ ), \mathbf{R}/\mathbf{Z})$. The assumption about $h^*(\ )$ implies that F has finitely generated homotopy groups. Let $E = DF$. Then $h^*(\ ) = \mathrm{Hom}(F_*(\ ), \mathbf{R}/\mathbf{Z}) = E\mathbf{R}/\mathbf{Z}_T^*(\ )$ by the previous corollary.

5. Applications.

DEFINITION 40. A homology operation is a natural transformation $E_n(\ ) \to E_m(\ )$ where $E_*(\ )$ is a generalized homology theory and n, m are fixed integers.

Notation 41. We will denote by $L(E, n, M)$ the set of such operations and by $\mathcal{O}_{a,c}(h^*(\ ), n, m)$ the set of additive, continuous cohomology operations $h^n(\ ) \to h^m(\ )$, where $h^*(\ )$ is a cohomology theory taking values in the category of topological groups.

LEMMA 42. (a) *All homology operations are additive.*

(b) *The Pontrjagin character functor*, c, *gives an isomorphism between* $L(E, n, m)$ *and* $\mathcal{O}_{a,c}(c(E_*(\ )), m, n)$ *for any spectrum* E.

PROOF. A proof of (a) may be found in Steenrod **[13]** while (b) follows from the fact that c is a duality.

Combining this lemma with Theorem 2 we have

THEOREM 43. *If* $E \in S_{fg}$ *then*:

(a) $L(E, n, m)$ *is isomorphic to* $\mathcal{O}_{a,c}(DE\mathbf{R}/\mathbf{Z}_T^*(\ ), m, n)$.

(b) $\phi \in \mathcal{O}_a(DE\mathbf{R}/\mathbf{Z}^*(\ ), m, n)$ *is continuous iff there exists an additive cohomology operation* ϕ *completing the following diagram*:

$$\begin{tikzcd} DE\mathbf{R}^m(\ ) \arrow[r, "\bar{\phi}"] \arrow[d, "(1\wedge p)^*"'] & DE\mathbf{R}^n(\ ) \arrow[d, "(1\wedge p)^*"'] \\ DE\mathbf{R}/\mathbf{Z}^m(\ ) \arrow[r, "\phi"] & DE\mathbf{R}/\mathbf{Z}^n(\ ). \end{tikzcd}$$

To actually construct continuous operations it is useful to turn to the locally rational case.

PROPOSITION 44. *If* $E \in S_{fg}$ *then*:

(a) $(EQ^*(\ ), EQ/\mathbf{Z}^*(\ ), (1 \wedge q)^*)$ *is a functor taking values in* C_{LR}.

(b) $EQ/\mathbf{Z}^*(\ )$ *has a topology making it a cohomology theory taking values in* AB_{LR}.

(c) *An additive cohomology operation* ϕ: $EQ/\mathbf{Z}^n(\ ) \to EQ/\mathbf{Z}^m(\ )$ *is continuous with respect to the topology of* (b) *iff there is an additive operation* ϕ *fitting into the diagram*

$$\begin{array}{ccc} EQ^n(\ ) & \to & EQ^m(\ ) \\ \downarrow(1\wedge q)^* & & \downarrow(1\wedge q)^* \\ EQ/\mathbf{Z}^n(\ ) & \to & EQ/\mathbf{Z}^m(\ ) \end{array}$$

PROOF. The proof of this theorem follows exactly the proof of the analogous euclidean result, with (a) corresponding to Proposition 36, (b) to Theorem 1, and (c) to Theorem 43.

THEOREM 45. *The functor S of Definition* 15 *induces an isomorphism*

$$\mathcal{O}_{a,c}(EQ/\mathbf{Z}_T^*(\ ), n, m) \simeq \mathcal{O}_{a,c}(E\mathbf{R}/\mathbf{Z}_T^*(\ ), n, m).$$

PROOF. This follows from the fact that S gives an equivalence of categories (Proposition 16).

A large supply of homology operations for $E_*(\ )$ may now be constructed from cohomology operations for $DE^*(\ )$. First, however, we must examine the method of introducing coefficients into a cohomology theory.

In §4 we constructed the spectrum $EQ/\mathbf{Z}$ by smashing with a Moore spectrum. For the construction of unstable cohomology operations this has the disadvantage that $EQ/\mathbf{Z}$ need not be an Ω-spectrum, even if E was. To circumvent this difficulty we recall another method of constructing $EQ/\mathbf{Z}$ due to Maunder [**9**].

For n a positive integer, let L_n denote the Moore space of type $(\mathbf{Z}/n\mathbf{Z}, 2)$, and for n, m positive integers with $n \mid m$, let $i_{n,m}$: $L_n \to L_m$ and $p_{n,m}$: $L_m \to L_n$ be maps whose effect in homology is the canonical injection and surjection, respectively. Maunder points out that L_n is self-5-dual in the sense of Spanier [**12**]. Thus for any space X we have a homotopy equivalence $X \wedge SL_n \simeq X^{SL_n}$.

If $\{E_i\}$ is an Ω-spectrum then $\{E_{i+2}^{SL_n}\}$ is an Ω spectrum equivalent to $\{E_{i\bar{2}} \wedge SL_n\} = E\mathbf{Z}/n\mathbf{Z}$. The maps $i_{n,m}$ and $p_{m,n}$ make $\{E_{i-2} \wedge SL_n\}$ and $\{E_{i+2}^{SL_n}\}$ into directed systems of spectra which are equivalent, since $i_{n,m}$ and $p_{m,n}$ are dual. Passing to the direct limit, we have an equivalence

$$EQ/\mathbf{Z} \simeq \left\{ \varinjlim_n E_{i-2} \wedge SL_n \right\} \simeq \left\{ \varinjlim E_{i+2}^{SL_n} \right\},$$

and the spectrum on the right is an Ω spectrum.

Since $\{E_i\}$ is an Ω-spectrum there is a 1-1 correspondence between unstable cohomology operations $E^k(\ ) \to E^l(\ )$ and homotopy classes of maps $E_k \to E_l$. Suppose ϕ': $E_{k+2} \to E_{l+2}$ is a map such that the corresponding operation is additive. For each n, ϕ' induces a map $E_{k+2}^{SL_n} \to E_{l+2}^{SL_n}$ by composition so, passing to the direct limit we obtain an unstable cohomology operation for $EQ/\mathbf{Z}$, ϕ, which is additive, since ϕ' is. In fact, using Proposition 44(c), we see that ϕ is actually in $\mathcal{O}_{a,c}(EQ/\mathbf{Z}_T^*(\ ), k, l)$. An operation ϕ: $EQ^k(\ ) \to EQ^l(\ )$ covering ϕ can be constructed from ϕ using the description of $(EQ)_i$ as $\varinjlim(E_{i+2}^{S_2})$.

A specific example of this construction is obtained by considering complex K-theory. Recall the following result due to D. W. Anderson [**2**]:

PROPOSITION 46. *If K, KO, KSp are spectra representing complex, real and symplectic K-theory, respectively, then $DK = K$, $DKO = KSp$, and $DKSp = KO$.*

Combining this with Theorem 43 we have

COROLLARY 47.

$$L(K,0,0) \simeq O_{a,c}(K\mathbf{R}/\mathbf{Z}_T^*(\ ),0,0) \simeq O_{a,c}(KQ/\mathbf{Z}_T^*(\ ),0,0).$$

Applying the construction described above to the Adams operations ψ^k, we have

COROLLARY 48. *There exist homology operations for complex K-theory dual to the Adams operations.*

PROOF OF THEOREM 4. The equivalence of (a) and (b) follows directly from Theorem 2, since $F \circ h^*(\ ) = E\mathbf{R}/\mathbf{Z}^*(\ )$ and, similarly, for h', E'.

The equivalence of (b), (c) and (d) is a standard homotopy argument.

PROOF OF COROLLARY 5. Let $f\colon S^m \to S^n$ be a map representing an element of order p (an odd prime) in $\Pi_m(S^n)$ with $m > n$. Let C_k be the cofibre of the map $k \cdot f$, $k = 1, 2, \ldots, p-1$. According to Artin and Mazur [3, p. 92] the spaces C_k are stably profinite homotopy equivalent, but not all homotopy equivalent. Taking suspension spectra of these spaces and applying Theorem 4, we obtain the result.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO N2L 3G1 CANADA